An asymptotic model for the propagation of long waves

with improved frequency dispersion

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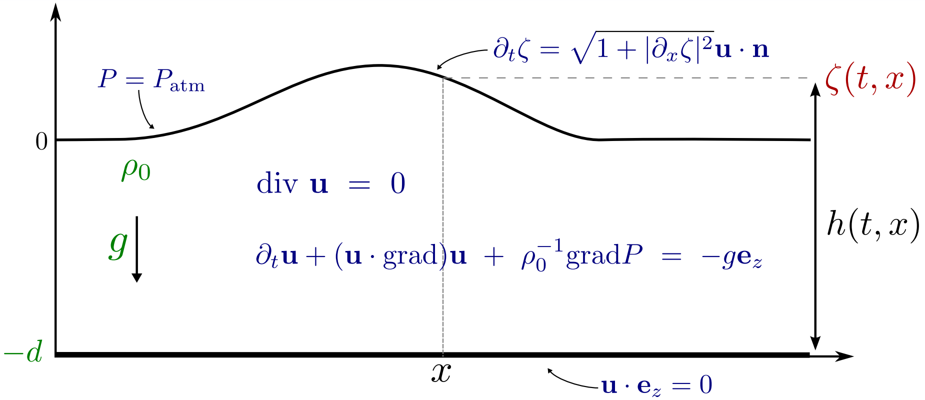
1. Motivation
   - The water-waves system
   - The Saint-Venant system
   - The Green-Naghdi system
   - Our modified GN system

2. Well-posedness
   - The Saint-Venant system
   - The (modified) Green-Naghdi system

3. Solitary waves
   - The Green-Naghdi system
   - The modified Green-Naghdi system

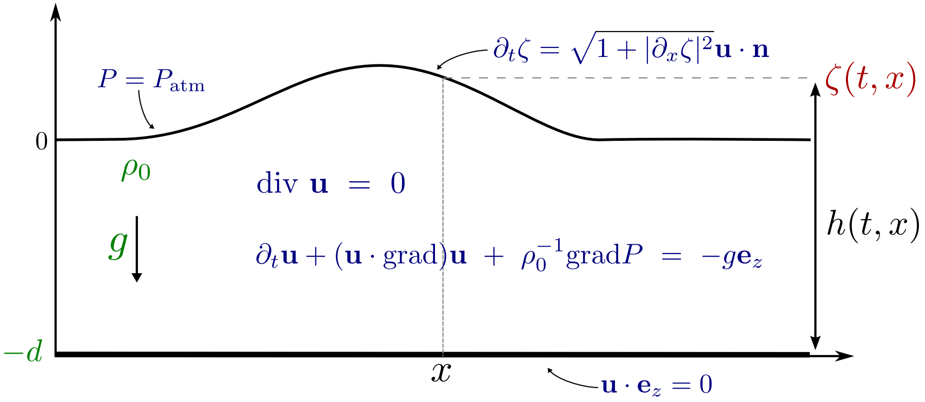
The water-waves system

The water-waves system



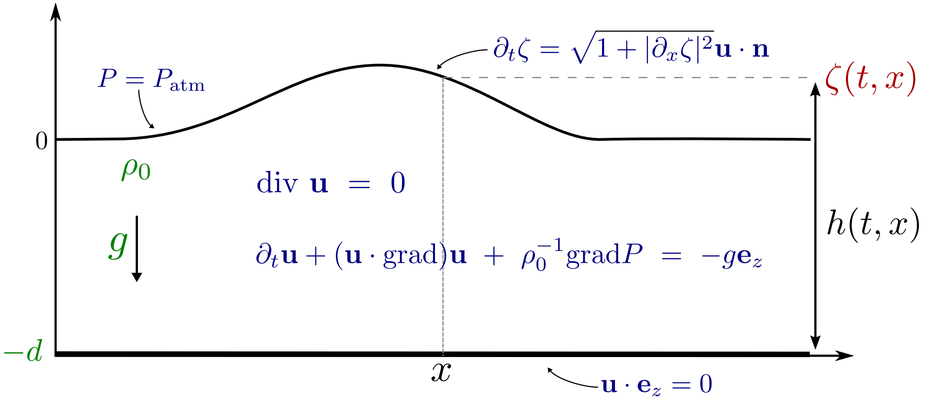
- The domain is an infinite layer with a free surface.
- The fluid is incompressible, the only external force is gravity.
- Particles of fluid cannot cross the surface or bottom.
- Surface tension, viscosity are not taken into account.

The water-waves system



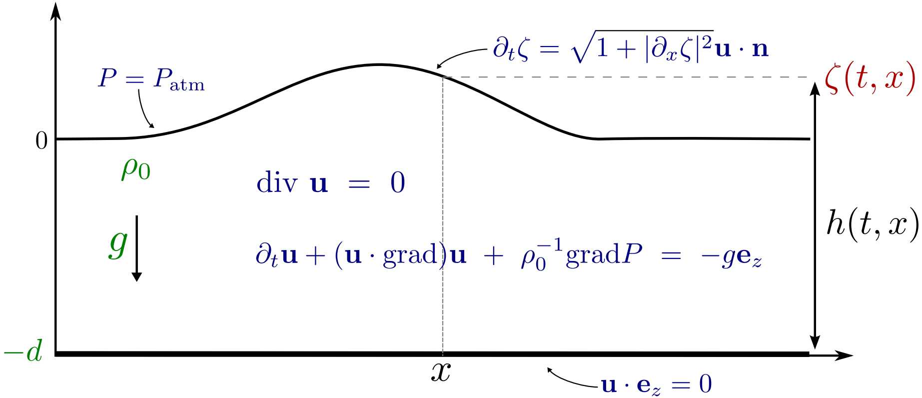
[Feynman] “[water waves] that are easily seen by everyone and which are usually used as an example of waves in elementary courses [...] are the worst possible example [...]; they have all the complications that waves can have.”

The Saint-Venant system



- Hydrostatic approximation : $\nabla P = g\mathbf{e}_z$
- Columnar motion : $(\overline{\mathbf{u}\cdot\mathbf{e}_x})^2 = \overline{(\mathbf{u}\cdot\mathbf{e}_x)^2}$
  (notation : $\bar{u}(x) = \frac{1}{d+\zeta}\int_{-d}^{\zeta} u(x,z)\mathrm{d}z$)
- Closed equations for variables ζ and $\overline{\mathbf{u}\cdot\mathbf{e}_x}$.

The Saint-Venant system



$$\begin{cases} \partial_t \zeta + \partial_x((d+\zeta)\overline{u}) = 0 \\ \partial_t \overline{u} + g\partial_x \zeta + \overline{u}\partial_x \overline{u} = 0 \end{cases} \qquad \text{(SV)}$$

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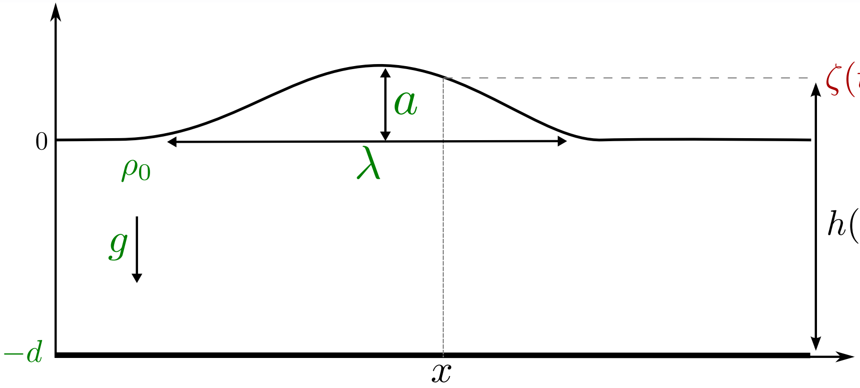

$$\epsilon \overset{\text{def}}{=} a/d \qquad ; \qquad \mu \overset{\text{def}}{=} d^2/\lambda^2.$$

$$\begin{cases} \partial_t \zeta + \partial_x((1+\epsilon\zeta)\overline{u}) = 0 \\ \partial_t \overline{u} + \partial_x \zeta + \epsilon \overline{u} \partial_x \overline{u} = \mathcal{O}(\mu) \end{cases} \tag{SV}$$

The Green-Naghdi system

[(c) Hitori Sushi (flickr)]

The Green-Naghdi system

$$\begin{cases} \partial_t \zeta + \partial_x (h\overline{u}) = 0 \\ \left(\mathrm{Id} + \mu \mathcal{T}[h]\right)\partial_t \overline{u} + \partial_x \zeta + \epsilon \overline{u} \partial_x \overline{u} + \epsilon\mu \mathcal{R}[h, \overline{u}] = \mathcal{O}(\mu^2) \end{cases} \quad \text{(GN)}$$

with

$$\mathcal{T}[h]V \stackrel{\text{def}}{=} -\frac{1}{3h}\partial_x (h^3 \partial_x V)$$
$$\mathcal{R}[h, \overline{u}] \stackrel{\text{def}}{=} -\frac{1}{3h}\partial_x \Big(h^3 \left(\overline{u}\partial_x^2 \overline{u} - (\partial_x \overline{u})^2 \right) \Big)$$

Formal derivation

[Serre'53, Su&Gardner'69, Green&Naghdi'76, Miles&Salmon'85...] [Bonneton&Lannes'09]

Properties

- Hamiltonian formulation (directly related to the water-waves system)
- Invariance with respect to horizontal/time translation, Galilean boost
- Conservation of mass, momentum, energy (Noether's theorem)

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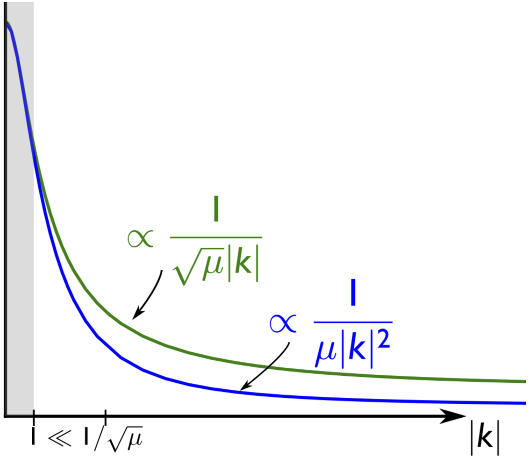
Dispersion relation

Explicit dispersion relation for plane waves $e^{i(kx-\omega(k)t)}$

$$\left(\frac{\omega(k)}{k}\right)^2 = \frac{\tanh(\sqrt{\mu}k)}{\sqrt{\mu}k} \quad \text{vs} \quad \left(\frac{\omega(k)}{k}\right)^2 = \frac{1}{1+\frac{\mu}{3}k^2}$$



Our modified Green-Naghdi system

$$\begin{cases} \partial_t\zeta + \partial_x(h\overline{u}) = 0 \\ \left(\mathrm{Id} + \mu\mathcal{T}^{\mathsf{F}}[h]\right)\partial_t\overline{u} + \partial_x\zeta + \epsilon\overline{u}\partial_x\overline{u} + \epsilon\mu\mathcal{R}^{\mathsf{F}}[h,\overline{u}] = \mathcal{O}(\epsilon\mu^2) \end{cases} \qquad \text{(mGN)}$$

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and $\mathsf{F} = F(\sqrt{\mu}|D|)$ i.e. (Fourier multiplier) $\widehat{\mathsf{F}u}(\xi) = F(\sqrt{\mu}|\xi|)\widehat{u}(\xi)$:

$$\mathsf{F} = \sqrt{\frac{3}{\sqrt{\mu}|D|\tanh(\sqrt{\mu}|D|)} - \frac{3}{\mu|D|^2}}$$

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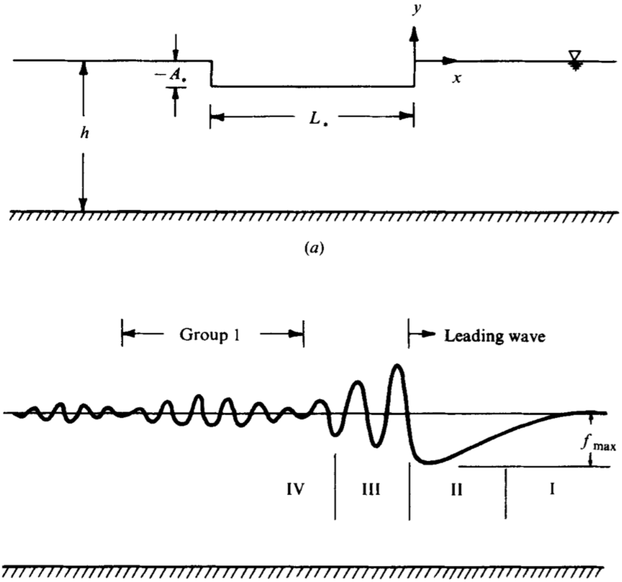
Experimental validation (?)



[Hammack&Segur '84]

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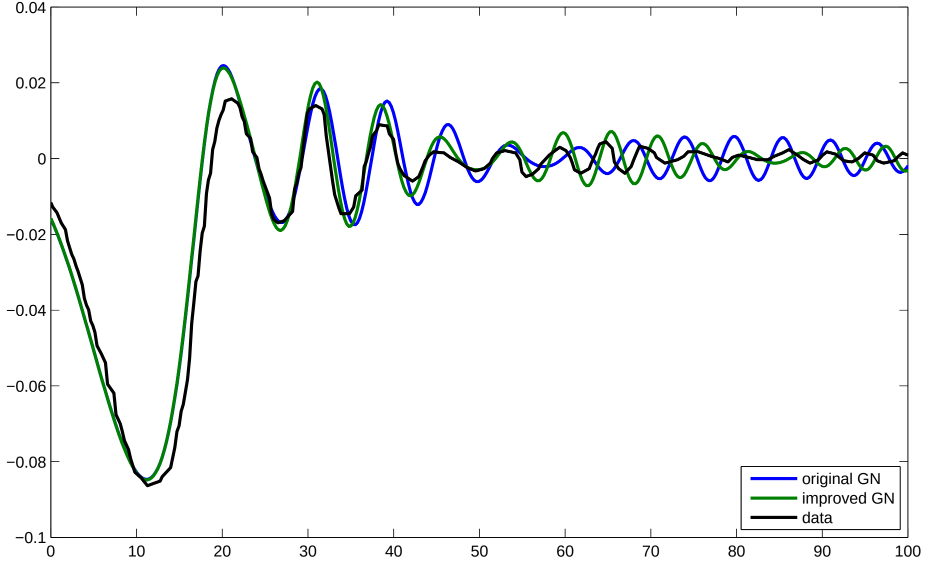

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The Saint-Venant system

We study the initial value problem for

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System of conservation laws (= compressible Euler equations).

Hyperbolic, symmetrizable $\Longrightarrow$ strong local well-posedness.

[Friedrichs, Garding, Lax, Leray, Kato] '50s, '60s

Let $\zeta^0, \overline{u}^0 \in H^s(\mathbb{R})^2$ with $s > 3/2$ be such that $1 + \epsilon\zeta^0 > 0$. Then there exists $T > 0$ and $(\rho, \overline{u}) \in C^0([0, T/\epsilon); H^s(\mathbb{R}^d)^{d+1})$ unique solution to the Saint-Venant system with initial data $\zeta^0, \overline{u}^0$.

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Sketch of the proof. We seek an a priori control in H^s of the solutions to

$$\partial_t \mathbf{u} + A(\mathbf{u})\partial_x \mathbf{u} = \mathbf{0}.$$

O.D.E. in Banach space $H^s(\mathbb{R}^d)$, but loss of derivatives?

Example: the solution to $\partial_t \mathbf{u} + A\partial_x \mathbf{u} = \mathbf{0}$ is $\mathbf{u} = e^{-tA\partial_x}\mathbf{u}^0$ with

$$\widehat{e^{-tA\partial_x}\mathbf{u}^0}(\xi) = e^{-itA\xi}\widehat{\mathbf{u}^0}(\xi).$$

thus $\left\| e^{-tA\partial_x} \right\|_{H^s \to H^s} \leq C$ iff A is diagonalizable with real eigenvalues (for instance if A is real symmetric, or if there exists S self-adjoint, positive definite such that SA is real symmetric)

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  Test the equation with $\mathbf{u}$ and integrate by parts:
  $$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} S(t,x)\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}} \big(\partial_x A(t,x) - \partial_t S(t,x)\big)\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}x.$$
  If $\partial_x A, \partial_t S \in L^\infty$, then (by Grönwall) $\big|\mathbf{u}\big|_{L^2} \lesssim \big|\mathbf{u}^0\big|_{L^2} e^{Ct}$.
  Differentiate the equation and use same trick $\Rightarrow \big|\mathbf{u}\big|_{H^n} \lesssim \big|\mathbf{u}^0\big|_{H^n} e^{Ct}$.
- The Picard iterates, defined by $S(\mathbf{u}^k)\partial_t \mathbf{u}^{k+1} + A(\mathbf{u}^k)\partial_x \mathbf{u}^{k+1} = \mathbf{0}$, converge (for T small) towards a solution of the nonlinear equation.

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may be symmetrized as follows

$$\begin{pmatrix} 1 & 0 \\ 0 & 1+\epsilon\zeta \end{pmatrix} \partial_t \begin{pmatrix} \zeta \\ \overline{u} \end{pmatrix} + \begin{pmatrix} \epsilon\overline{u} & 1+\epsilon\zeta \\ 1+\epsilon\zeta & (1+\epsilon\zeta)\epsilon\overline{u} \end{pmatrix} \partial_x \begin{pmatrix} \zeta \\ \overline{u} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

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Existence and uniqueness of a strong solution to the Cauchy problem, in the Sobolev setting and uniformly with respect to $\mu \ll 1$.

- Green-Naghdi system [Li'02]
- modified GN system (with surface tension) [Duchene,Israwi&Talhouk'16]

Difficulties

- The weak dispersion ($\mu \ll 1$) forbids the use of dispersive techniques.
- The presence of higher-order operators makes it difficult to control derivatives of the unknown, because the commutator

$$\big[\partial_x, \mathcal{R}[\underline{h}, \underline{\overline{u}}]\big] \text{ is not of order zero.}$$

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Energy space : Provided $\underline{h} \in L^\infty$ and $\underline{h} > 0$, one has

$$\int_{\mathbb{R}} \zeta^2 + \big(\underline{h}\overline{u} + \mu\mathcal{T}^{\mathsf{F}}[\underline{h}]\overline{u}\big)\overline{u}\,\mathrm{d}x \approx \|\zeta\|_{L^2}^2 + \|\overline{u}\|_{L^2}^2 + \mu\|\partial_x\mathsf{F}\overline{u}\|_{L^2}^2 \stackrel{\text{def}}{=} \|\zeta\|_{L^2}^2 + \|\overline{u}\|_{X^0}^2$$

Quasilinearisation : for n sufficiently large,

$$\begin{cases} \partial_t\zeta^{(n)} + \epsilon\overline{u}\partial_x\zeta^{(n)} + h\partial_x\overline{u}^{(n)} = f_1 \\ h\big(\mathrm{Id} + \mu\mathcal{T}^{\mathsf{F}}[h]\big)\partial_t\overline{u}^{(n)} + h\partial_x\zeta^{(n)} + \epsilon h\overline{u}\partial_x\overline{u}^{(n)} - \frac{\epsilon\mu}{3}h^3\overline{u}\partial_x(\partial_x\mathsf{F})^2\overline{u}^{(n)} = f_2 \end{cases}$$

with $\|f_1\|_{L^2} \lesssim \epsilon C(\|\zeta\|_{H^n}, \|\overline{u}\|_{X^n})$ and $\|f_2\|_{(X^0)'} \lesssim \epsilon C(\|\zeta\|_{H^n}, \|\overline{u}\|_{X^n})$.

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$$\begin{cases} \partial_t\zeta + \partial_x(h\overline{u}) = 0 \\ h\big(\mathrm{Id} + \mu\mathcal{T}^{\mathsf{F}}[h]\big)\partial_t\overline{u} + h\partial_x\zeta + \epsilon h\overline{u}\partial_x\overline{u} + \epsilon\mu h\mathcal{R}^{\mathsf{F}}[h,\overline{u}] = 0 \end{cases} \qquad \text{(mGN)}$$

$$h\mathcal{T}^{\mathsf{F}}[h]V \overset{\text{def}}{=} -\frac{1}{3}\partial_x(\mathsf{F}h^3\partial_x\mathsf{F}V)$$

$$h\mathcal{R}^{\mathsf{F}}[h,\overline{u}] \overset{\text{def}}{=} -\frac{1}{3}\partial_x\Big(h^3\big(\overline{u}(\partial_x\mathsf{F})^2\overline{u} - (\partial_x\mathsf{F}\overline{u})^2\big)\Big)$$

Energy space : Provided $\underline{h} \in L^\infty$ and $\underline{h} > 0$, one has

$$\int_{\mathbb{R}} \zeta^2 + \big(\underline{h}\overline{u} + \mu\mathcal{T}^{\mathsf{F}}[\underline{h}]\overline{u}\big)\overline{u}\,\mathrm{d}x \approx \big\|\zeta\big\|_{L^2}^2 + \big\|\overline{u}\big\|_{L^2}^2 + \mu\big\|\partial_x\mathsf{F}\overline{u}\big\|_{L^2}^2 \overset{\text{def}}{=} \big\|\zeta\big\|_{L^2}^2 + \big\|\overline{u}\big\|_{X^0}^2$$

Quasilinearisation : for n sufficiently large,

$$\begin{cases} \partial_t\zeta^{(n)} + \epsilon\overline{u}\partial_x\zeta^{(n)} + h\partial_x\overline{u}^{(n)} = f_1 \\ h\big(\mathrm{Id} + \mu\mathcal{T}^{\mathsf{F}}[h]\big)\partial_t\overline{u}^{(n)} + h\partial_x\zeta^{(n)} + \epsilon h\overline{u}\partial_x\overline{u}^{(n)} - \frac{\epsilon\mu}{3}h^3\overline{u}\partial_x(\partial_x\mathsf{F})^2\overline{u}^{(n)} = f_2 \end{cases}$$

with $\big\|f_1\big\|_{L^2} \lesssim \epsilon C(\big\|\zeta\big\|_{H^n}, \big\|\overline{u}\big\|_{X^n})$ and $\big\|f_2\big\|_{(X^0)'} \lesssim \epsilon C(\big\|\zeta\big\|_{H^n}, \big\|\overline{u}\big\|_{X^n})$.

The (modified) Green-Naghdi system

$$\begin{cases} \partial_t \zeta + \partial_x (h\overline{u}) = 0 \\ h\big(\mathrm{Id} + \mu \mathcal{T}^{\mathsf{F}}[h]\big)\partial_t \overline{u} + h\partial_x \zeta + \epsilon h \overline{u} \partial_x \overline{u} + \epsilon\mu h \mathcal{R}^{\mathsf{F}}[h, \overline{u}] = 0 \end{cases} \quad \text{(mGN)}$$

Main result : Existence and uniqueness of a strong solution

Assume the Fourier multiplier $\mathsf{F} = F(\sqrt{\mu}D)$ is such that

1. F is even and non-negative;
2. $\xi \mapsto |\xi| F(\xi)$ is sub-additive.

Let $(\zeta^0, \overline{u}^0) \in (H^n \times X^n)$ with n sufficiently large be such that $1 + \epsilon\zeta^0 > 0$. Then there exists $T > 0$ and $(\zeta, \overline{u}) \in C^0([0, T/\epsilon); H^n \times X^n)$ unique strong solution to (mGN) with initial data $\zeta^0, \overline{u}^0$.

Bonus : By [Lannes], and if $F \in W^{2,\infty}$, and $F(0) = 1$, then the solution to the water-waves system with corresponding initial data remains close at precision $\mathcal{O}(\mu^2)$ over the time interval $[0, T/\epsilon)$.

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   - The modified Green-Naghdi system

The minimization problem

$$\begin{cases} \partial_t \zeta + \partial_x (h\overline{u}) = 0 \\ \big(\mathrm{Id} + \mu \mathcal{T}[h]\big)\partial_t \overline{u} + \partial_x \zeta + \epsilon \overline{u} \partial_x \overline{u} + \epsilon\mu \mathcal{R}[h,\overline{u}] = \mathcal{O}(\mu^2) \end{cases} \tag{GN}$$

Hamiltonian structure

$$\begin{cases} \partial_t \zeta + \partial_x \frac{\delta \mathcal{H}}{\delta v} = 0 \\ \partial_t v + \partial_x \frac{\delta \mathcal{H}}{\delta \zeta} = 0 \end{cases}$$

where

$$v \stackrel{\text{def}}{=} \big(\mathrm{Id} + \mu \mathcal{T}[h]\big) u$$

and

$$\mathcal{H}(\zeta, v) \stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}} \zeta^2 + v \big(\mathrm{Id} + \mu \mathcal{T}[h]\big)^{-1} v \, \mathrm{d}x$$

Preserved quantities

$$\int_{\mathbb{R}} \zeta \quad ; \quad \int_{\mathbb{R}} v \quad ; \quad \mathcal{I}(\zeta, v) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \zeta v \, \mathrm{d}x \quad \text{and} \quad \mathcal{H}(\zeta, v).$$

The minimization problem

$$\begin{cases} \partial_t \zeta + \partial_x (h\overline{u}) = 0 \\ \big(\mathrm{Id} + \mu \mathcal{T}[h]\big)\partial_t \overline{u} + \partial_x \zeta + \epsilon \overline{u} \partial_x \overline{u} + \epsilon\mu \mathcal{R}[h, \overline{u}] = \mathcal{O}(\mu^2) \end{cases} \tag{GN}$$

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Preserved quantities

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Minimization problem

Solitary waves satisfy $\delta \mathcal{H} - c \delta \mathcal{I} = 0$, but critical points of $\mathcal{H} - c\mathcal{I}$ are not minimizers or maximizers.

Solitary waves satisfy $\delta \mathcal{E}(\zeta) = 2c^{-2}\zeta$ where $\mathcal{E}(\zeta) = \mathcal{I}\big(\zeta\, ,\, (\mathrm{Id} + \mu \mathcal{T}[h])\zeta\big)$.

We seek $\boxed{\arg\min\{\mathcal{E}(\zeta) \ : \ \|\zeta\|_{H^1} \leq 1, \|\zeta\|_{L^2}^2 = q\}.}$

The strategy

We seek $\boxed{\arg\min\{\mathcal{E}(\zeta) \ : \ \|\zeta\|_{H^1} \leq 1, \|\zeta\|_{L^2}^2 = q\}}$
where (setting $\epsilon = \mu = 1$)

$$\mathcal{E}(\zeta) = \int_{\mathbb{R}} \frac{\zeta^2}{1+\zeta} + \frac{1}{3}(1+\zeta)^3 \partial_x \Big(\frac{\zeta}{1+\zeta}\Big)^2 \mathrm{d}x.$$

Consider a minimizing sequence, and try to prove that it “converges”.

The strategy

Lions' concentration-compactness argument

Any sequence $\{e_n\}_{n\in\mathbb{N}} \subset L^1(\mathbb{R})$ of non-negative functions such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} e_n \, \mathrm{d}x = I > 0$$

admits a subsequence for which one of the following phenomena occurs.

- (Vanishing) For each $r > 0$, one has $\lim_{n\to\infty} \left(\sup_{x\in\mathbb{R}} \int_{x-r}^{x+r} e_n \, \mathrm{d}x \right) = 0$.
- (Dichotomy) There are real sequences $\{x_n\}_{n\in\mathbb{N}}, \{M_n\}_{n\in\mathbb{N}}, \{N_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $I^* \in (0, I)$ such that $M_n, N_n \to \infty$, $M_n/N_n \to 0$, and

$$\int_{x_n-M_n}^{x_n+M_n} e_n \, \mathrm{d}x \to I^* \quad \text{and} \quad \int_{x_n-N_n}^{x_n+N_n} e_n \, \mathrm{d}x \to I^* \qquad \text{as } n \to \infty.$$

- (Concentration) There exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ with the property that for each $\epsilon > 0$, there exists $r > 0$ with

$$\int_{x_n-r}^{x_n+r} e_n \, \mathrm{d}x \geq I - \epsilon \qquad \text{for all } n \in \mathbb{N}.$$

The strategy

We seek $\boxed{\arg\min\{\mathcal{E}(\zeta) \;:\; \|\zeta\|_{H^1} \leq 1, \|\zeta\|_{L^2}^2 = q\}}$
where (setting $\epsilon = \mu = 1$)

$$\mathcal{E}(\zeta) = \int_{\mathbb{R}} \frac{\zeta^2}{1+\zeta} + \frac{1}{3}(1+\zeta)^3 \partial_x\Big(\frac{\zeta}{1+\zeta}\Big)^2 \mathrm{d}x.$$

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Coercivity

If q sufficiently small, $\|\zeta\|_{L^\infty} < 1$ and $\mathcal{E}(\zeta) \approx \|\zeta\|_{H^1}^2$.

Excluding Dichotomy

We need to exclude the Vanishing scenario (easy) and Dichotomy scenario.

Claim : Sub-homogeneity and sub-additivity

If q sufficiently small, $q \mapsto I_q = \min\{\mathcal{E}(\zeta) \ : \ \|\zeta\|_{H^1} \leq 1, \|\zeta\|_{L^2}^2 = q\}$ is sub-homogeneous ($I_{aq} < aI_q$) and thus sub-additive ($I_{q_1+q_2} < I_{q_1} + I_{q_2}$).

We shall use the three following results :

Coercivity

If q sufficiently small, $\|\zeta\|_{L^\infty} < 1$ and $\mathcal{E}(\zeta) \approx \|\zeta\|_{H^1}^2$.

Expansion (long waves)

$$\mathcal{E}(\zeta) = \int_{\mathbb{R}} \zeta^2 - \zeta^3 + \frac{1}{3}\zeta_x^2 dx + \mathcal{O}\left(\|\zeta\|_{L^\infty}^2 \|\zeta\|_{L^2}^2 + \|\zeta\|_{L^\infty} \|\zeta_x\|_{L^2}^2 + \|\zeta_x\|_{L^2} \|\zeta_{xxx}\|_{L^2}\right)$$

Expansion (small waves)

$$\mathcal{E}(\zeta) = \int \zeta^2 + \frac{1}{[illegible]}\zeta^2 [illegible] \zeta^3 [illegible] \frac{1}{[illegible]}\zeta\zeta^2 dx + \mathcal{O}\left(\|\zeta\|^2 [illegible] \|\zeta\|^2 [illegible]\right)$$

Excluding Dichotomy

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Step 1

Expansion (long waves)

$$\mathcal{E}(\zeta) = \int_{\mathbb{R}} \zeta^2 - \zeta^3 + \frac{1}{3}\zeta_x^2 \mathrm{d}x + \mathcal{O}\left(\|\zeta\|_{L^\infty}^2 \|\zeta\|_{L^2}^2 + \|\zeta\|_{L^\infty} \|\zeta_x\|_{L^2}^2 + \|\zeta_x\|_{L^2}^2 \|\zeta_{xxx}\|_{L^2} \right)$$

Corollary

If q sufficiently small, $I_q < q - mq^{5/3}$ with $m > 0$.

Let $\psi \in C_c^\infty$ such that $\psi \geq 0$ and $\|\psi\|_{L^2}^2 = 1$.

1. For λ sufficiently small, $\psi_\lambda = \lambda^{1/2}\psi(\lambda\cdot)$ satisfies $\psi_\lambda^3 - {\psi_\lambda}_x^2 = 2m > 0$;
2. $\phi_q = q^{2/3}\phi_q(q^{1/3}\cdot)$ satisfies $\mathcal{E}(\phi_q) = q - 2mq^{5/3} + \mathcal{O}(q^2)$.

Coercivity

If q sufficiently small, $\|\zeta\|_{L^\infty} < 1$ and $\mathcal{E}(\zeta) \approx \|\zeta\|_{H^1}^2$.

Corollary

If q sufficiently small and (ζ_n) a minimizing sequence, $\|\zeta_n\|_{H^1}^2 \leq Cq < 1$.

Step 1

Expansion (long waves)

$$\mathcal{E}(\zeta) = \int_{\mathbb{R}} \zeta^2 - \zeta^3 + \frac{1}{3}\zeta_x^2 \mathrm{d}x + \mathcal{O}\left(\|\zeta\|_{L^\infty}^2 \|\zeta\|_{L^2}^2 + \|\zeta\|_{L^\infty} \|\zeta_x\|_{L^2}^2 + \|\zeta_x\|_{L^2}^2 \|\zeta_{xxx}\|_{L^2} \right)$$

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If q sufficiently small and (ζ_n) a minimizing sequence, $\|\zeta_n\|_{H^1}^2 \leq Cq < 1$.

Step 2

Expansion (small waves)

$$\mathcal{E}(\zeta) = \int_{\mathbb{R}} \zeta^2 + \frac{1}{3}\zeta_x^2 - \zeta^3 - \frac{1}{3}\zeta\zeta_x^2 \mathrm{d}x + \mathcal{O}\left(\|\zeta\|_{L^\infty}^2 \|\zeta\|_{H^1}^2\right)$$

Corollary

If q sufficiently small and $a \in (1, a_0]$, $I_{aq} < aI_q$.

For (ζ_n) a minimizing sequence,

$$I_{aq} \leq \mathcal{E}(a^{1/2}\zeta_n) = a\mathcal{E}(\zeta_n) - (a^{3/2} - a)\int_{\mathbb{R}} \zeta_n^3 + \frac{1}{3}\zeta_n{\zeta_n}_x^2 \,\mathrm{d}x + \mathcal{O}((a^{3/2} - a)q^2).$$

and

$$-(\zeta_n^3 + \frac{1}{3}\zeta_n{\zeta_n}_x^2) = \mathcal{E}(\zeta_n) - \int_{\mathbb{R}} \zeta_n^2 + \frac{1}{3}{\zeta_n}_x^2 \,\mathrm{d}x + \mathcal{O}(q^2) < -\frac{1}{2}mq^{5/3}.$$

Taking the limit as $n \to \infty$, we find

$$I_{aq} \leq aI_q - (a^{3/2} - a)mq^{5/3} < aI_q.$$

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We seek $\boxed{\arg\min\{\mathcal{E}(\zeta) \ : \ \|\zeta\|_{H^\nu} \leq 1, \|\zeta\|_{L^2}^2 = q\}}$
where

$$\mathcal{E}(\zeta) = \int_{\mathbb{R}} \frac{\zeta^2}{1+\zeta} + \frac{1}{3}(1+\zeta)^3 \partial_x \mathsf{F}\Big(\frac{\zeta}{1+\zeta}\Big)^2 \mathrm{d}x, \qquad \mathsf{F} \approx \frac{1}{1+|D|^{1/2}}$$

Difficulties (similar as [Ehrnström, Groves & Wahlén '12])

1. $\partial_x \mathsf{F}$ is a nonlocal operator
2. F is a (1/2-) smoothing operator

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Difficulties (similar as [Ehrnström, Groves & Wahlén '12])

1. $\partial_x \mathsf{F}$ is a nonlocal operator
   Not a big deal. If ζ has compact support and x is outside the support, then for any $j \geq 2$,

$$|\partial_x \mathsf{F}\zeta|(x) \leq \frac{C_j}{\mathrm{dist}(x, \mathrm{supp}\zeta)^j} \|\zeta\|_{L^2}$$

   (using $\partial_\xi^j(\xi F(\xi)) \in L^2$)
2. F is a (1/2-) smoothing operator

The minimization problem

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Difficulties (similar as [Ehrnström, Groves & Wahlén '12])

1. $\partial_x \mathsf{F}$ is a nonlocal operator
2. F is a (1/2-) smoothing operator
   More problematic. How to prove that the minimizing sequence (ζ_n) satisfies $\|\zeta_n\|_{H^\nu} \lesssim q$?

A special minimizing sequence

Pb : Prove that a minimizing sequence (ζ_n) satisfies $\|\zeta_n\|_{H^\nu} \lesssim q$
Note that solutions of the Euler-Lagrange equation

$$2\frac{\zeta}{1+\zeta} - \frac{\zeta^2}{(1+\zeta)^2} - \frac{2}{3}\frac{1}{(1+\zeta)^2}\partial_x \mathsf{F}\{(1+\zeta)^3\partial_x \mathsf{F}\{\frac{\zeta}{1+\zeta}\}\} + \left(\left((1+\zeta)\partial_x \mathsf{F}\{\frac{\zeta}{1+\zeta}\}\right)^2\right)^2 + 2\alpha\zeta = 0.$$

satisfies the estimate (but this the solution we seek).

Solution :

- Consider the problem on $\mathbb{T}$ with a penalization
  $\arg\min\{\mathcal{E}_P(\zeta) + \varrho(\|\zeta\|_{H^\nu_P}) \ : \ \|\zeta\|_{H^\nu} \leq 1, \|\zeta\|_{L^2}^2 = q\}$
  $\rightsquigarrow$ The solution solves an Euler equation, and thus $\|\zeta\|_{H^\nu_P} \lesssim q$.
- Let the period P_n go to infinity.
  $\rightsquigarrow$ allows to construct a minimizing sequence satisfying $\|\zeta_n\|_{H^\nu} \lesssim q$.

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- Let the period P_n go to infinity.
  $\rightsquigarrow$ allows to construct a minimizing sequence satisfying $\big\|\zeta_n\big\|_{H^\nu} \lesssim q$.

The result

Main result [VD, Nilsson & Wahlén]

Let F admissible: sufficiently smooth and decaying as $(1+|\xi|)^{-\theta}$, $\theta \in [0.1)$; and set $\nu > 1/2$ such that $\nu \geq 1-\theta$. Let D_q be the set of minimizers of $\mathcal{E}$ over $\{\zeta \ : \ \|\zeta\|_{H^\nu} \leq 1, \|\zeta\|_{L^2}^2 = q\}$. Then there exists $q_0 > 0$ such that for all $q \in (0, q_0)$, the following statements hold:

- The set D_q is nonempty and each element in D_q solves the traveling wave equation, which yields a supercritical solitary wave solution.
- For any minimizing sequence $(\zeta_n)_{n\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}}\|\zeta_n\|_{H^\nu} < 1$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers such that a subsequence of $(\zeta_n(\cdot + x_n))_{n\in\mathbb{N}}$ converges to an element in D_q.
- There exist constants $m, M_n > 0$ such that

$$\forall n \in \mathbb{N}, \qquad \|\zeta\|_{H^\nu(\mathbb{R})}^2 \leq M_n q \quad \text{and} \quad c^{-2} \leq 1 - m q^{\frac{2}{3}},$$

uniformly over $q \in (0, q_0)$ and $\zeta \in D_q$.

Thank you for your attention